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Topological entropy and $\mathscr{P}$-chaos of a coupled lattice system with non-zero coupling constant related with Belusov–Zhabotinskii reaction

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Abstract This paper focuses on the chaotic properties of the following systems stated by Kaneko (Phys Rev Lett 65:1391–1394, 1990) which is related to the Belusov–Zhabotinskii reaction:

$$x_n^{m+1} = (1-\varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m) \right],$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in (0, 1]$ is coupling constant and f is a continuous selfmap of [0, 1]. It is shown that for every continuous selfmap f of the interval [0, 1], the topological entropy of such a coupled lattice system is not less than the topological entropy of the map f, and that for every continuous selfmap of the interval [0, 1] with positive topological entropy, such a system is $\mathscr{P}$-chaotic, where $\mathscr{P}$ denotes one of the three properties: Li–Yorke chaos,

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distributional chaos, ω-chaos. These results extend the ones of Wu and Zhu (J Math Chem 50:1304–1308, 2012), (J Math Chem 50:2439–2445, 2012) and Li et al. (J Math Chem 51:1712–1719, 2013).

Keywords Coupled map lattice · $\mathscr{P}$-chaos · Topological entropy

Mathematics Subject Classification 54H20 · 58F03 · 47A16

1 Introduction

A *dynamical system* is a pair (X, f), where X is a compact metric space and f is a continuous selfmap on X. Sharkovskii's amazing discovery [18], as well as Li and Yorke's famous work which introduced the concept of 'chaos' in a mathematically rigorous way [14], have activated sustained scientific interest and provoked the recent rapid development of the frontier research on discrete chaos theory (see, e.g., [2–6,15,17]). An essential feature of chaos is the impossibility of prediction of its dynamics due to the exponential separation of any two close orbits. According to the idea of Li and Yorke [14], a subset $D \subset X$ is called a scrambled set for f, if every pair x, y from D with $x \neq y$ is proximal and not asymptotic, i.e.

$$\liminf_{n\to\infty} d\left(f^n(x), f^n(y)\right) = 0, \quad \text{and} \quad \limsup_{n\to\infty} d\left(f^n(x), f^n(y)\right) > 0.$$

If a scrambled set D of f is uncountable, then it is called a Li–Yorke chaotic set for f, and f is said to be chaotic in the sense of Li–Yorke.

In many physical/ chemical engineering applications such as digital filtering, imaging and spatial dynamical system, dynamical systems have recently appeared as an important subject for investigation (see e.g., [4,6–11,16,19]). For example, Guirao and Lampart [7,8] studied a lattice dynamical system with coupling constant $\varepsilon = 0$ related with Belusov–Zhabotinskii reaction and proved that it is chaotic in the sense of Li–Yorke and chaotic in the sense of Devaney, and has positive entropy. Very recently, Wu and Zhu [20,21] considered and studied the chaotic properties of the following coupled lattice system:

$$x_n^{m+1} = (1-\varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m)\right], \tag{1.1}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in [0, 1]$ is coupling constant and f is a continuous selfmap on I. Also, they proved that this system with non-zero coupling constant is chaotic in the sense of Li–Yorke and distributional (p, q)-chaos for any pair $0 \leq p \leq q \leq 1$ and the tent map. Inspired by the above results, our purpose here is to continue to study topological chaos and $\mathscr{P}$-chaos of this lattice system with non-zero coupling constant by further developing the results in [12,20,21]. In particular, we prove that for every continuous selfmap f of the interval $[0, 1]$, the topological entropy of such a coupled lattice system is not less than the topological entropy of the map f, and that for every continuous selfmap of the interval [0, 1] with positive topological entropy, such a system is $\mathscr{P}$-chaotic, where $\mathscr{P}$ denotes one of the three properties: Li–Yorke chaos, distributional chaos, ω-chaos. Our results extend the ones of Wu and Zhu [20,21] and Li et al. [12].

2 Preliminaries

First, Let us recall some important notions of chaos such as distributional chaos, ω-chaos and positive topological entropy. Throughout this paper, let I denote the unite closed interval [0, 1] and $\mathscr{P}$ denote one of the three properties: Li–Yorke chaos, distributional chaos, ω-chaos.

A generalization of the concept of Li–Yorke chaos is *distributional chaos*, introduced by Schweizer and Smítal [17].

Let (X, f) be a dynamical system. For any pair of points $x, y \in X$ and any $n \in \mathbb{N}$, let

$$\xi(x, y, t, n) = \left|\{j \in \mathbb{N} : d(f^j(x), f^j(y)) < t, 1 \le j \le n\}\right|,$$

where $|A|$ denotes the cardinality of set A. Define *lower* and *upper distributional functions* $\mathbb{R} \longrightarrow [0, 1]$ generated by f, x and y, as

$$F_{x,y}(t, f) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n),$$

and

$$F^*_{x,y}(t, f) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, t, n),$$

respectively.

Definition 2.1 A dynamical system (X, f) is distributionally chaotic if there exists an uncountable subset $D \subset X$ such that for any pair of distinct points $x, y \in D$, one has $F^*_{x,y}(t, f) = 1$ for all $t > 0$ and $F_{x,y}(\varepsilon, f) = 0$ for some $\varepsilon > 0$. The set D is a distributionally chaotic set and the pair (x, y) a distributionally chaotic pair.

Definition 2.2 Two dynamical system (X, f) and (Y, g) are topologically conjugated if there exists a homeomorphism $h : X \longrightarrow Y$ such that $h \circ f = g \circ h$, and the homeomorphism h is called a conjugacy.

An attempt to measure the complexity of a dynamical system is based on a computation of how many points are necessary in order to approximate (in some sense) with their orbits all possible orbits of the system. A formalization of this intuition leads to the notion of topological entropy of the map f, which is due to Adler et al. [1]. We recall here the equivalent definition formulated by Bowen [3], and independently by Dinaburg [6]: the topological entropy of a map f is a number $h_{top}(f) \in [0, +\infty]$ defined by

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} |E(n, f, \epsilon)|$$

where $E(n, f, \epsilon)$ is a (n, f, ϵ)-span with minimal possible number of points.

Definition 2.3 A dynamical system (X, f) is topologically chaotic if its topological entropy $h_{top}(f)$ is positive.

For a dynamical system (X, f), the set of periodic points of f and the limit set of a point x, which consists of all accumulation points of the orbit of x are denoted by $\mathrm{Per}(f)$ and $\omega_f(x)$, respectively. A subset $D \subset X$ is said to be a *ω-scrambled set for* f, if for any pair of distinct points $x, y \in D$, the following conditions are satisfied:

(1) $\omega_f(x) \setminus \omega_f(y)$ is uncountable;
(2) $\omega_f(x) \cap \omega_f(y) \neq \emptyset$;
(3) $\omega_f(x) \setminus \mathrm{Per}(f)$ is uncountable.

Definition 2.4 (see [13]) A dynamical system (X, f) is ω-chaotic if there exists an uncountable ω-scrambled set for f.

Proposition 2.1 *If* (X, f) *and* (Y, g) *are topologically conjugated systems, then* $h_{top}(f) = h_{top}(g)$.

Proposition 2.2 (see [15]) *If a dynamical system* (I, f) *is topologically chaotic, then there exists an invariant distributional chaotic set for* f.

Proposition 2.3 (see [13,17]) *A dynamical system* (I, f) *is topologically chaotic if and only if it is distributionally chaotic if and only if it is ω-chaotic.*

3 Main results

The state space of lattice dynamical system (LDS) is the set

$$\mathcal{X} = \{x : x = \{x_i\}, x_i \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|x_i\| < \infty\}.$$

where $a \geq 1$ is the dimension of the range space of the map of state x_i, $b \geq 1$ is the dimension of the lattice and the l^2 norm

$$\|x\|_2 = \left(\sum_{i \in \mathbb{Z}^b} |x_i|^2 \right)^{\frac{1}{2}}$$

is usually taken ($|x_i|$ is the length of the vector x_i).

Now we deal with the system (1.1) which is related to the Belusov–Zhabotinskii reaction

In general, one of the following periodic boundary conditions of the system (1.1) is assumed:

$$x_n^m = x_{n+L}^m,$$
$$x_n^m = x_n^{m+L},$$
$$x_n^m = x_{n+L}^{m+L},$$

standardly, the first case of the boundary conditions is used.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L)) = \left(\sum_{i=1}^{L} (x_i - y_i)^2\right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L) \in I^L$.

Let the map $F : (I^L, d) \to (I^L, d)$ be defined by $F(x_1, x_2, \ldots, x_L) = (y_1, y_2, \ldots, y_L)$ where $y_i = (1 - \varepsilon) f(x_i) + \frac{\varepsilon}{2}(f(x_{i-1}) + f(x_{i+1}))$. We note that the system (1.1) is equivalent to the system (I^L, F).

Motivated by [12,20,21], we have the following results.

Theorem 3.1 *For any* $0 < \varepsilon \leq 1$ *and any continuous selfmap* f *on* $[0, 1]$*, the topological entropy of the system* (1.1) *is not less than* $h_{top}(f)$.

Proof Denote $\Delta = \left\{(x_1, \ldots, x_L) \in I^L : x_1 = \cdots = x_L\right\}$. It can be easily verified that Δ is a closed invariant subset of F.

Now we assert that $(\Delta, F|_\Delta)$ and (I, f) are topologically conjugated.

Indeed, define $h : \Delta \longrightarrow I$ by

$$h(x, \ldots, x) = x,$$

for any $(x, \ldots, x) \in \Delta$. Clearly, h is a homeomorphism.

Given any $(x, \ldots, x) \in \Delta$, we have

$$h \circ F((x, \ldots, x)) = h((f(x), \ldots, f(x))) = f(x),$$

and

$$f \circ h((x, \ldots, x)) = f(x).$$

This implies that $h \circ F = f \circ h$. i.e., $(\Delta, F|_\Delta)$ and (I, f) are topologically conjugated. This together with Proposition 2.1 implies that

$$h_{top}(F) \geq h_{top}(F|_\Delta) = h_{top}(f).$$

Thus, the proof is completed. □

Corollary 3.1 *For any* $0 < \varepsilon \leq 1$ *and any unimodal map* f*, the topological entropy of the system* (1.1) *is not less than* $\log 2$.

Theorem 3.2 *For any* $0 < \varepsilon \leq 1$ *and any continuous selfmap* f *on* $[0, 1]$ *with positive topological entropy, the system* (1.1) *has an invariant distributional chaotic subset.*

Proof Applying Proposition 2.2, let D be an invariant distributional chaotic subset of f. Similarly to the proof of [21, Theorem1], it is not difficult to check that the set $\{(x_1, x_2, \ldots, x_L) \in I^L : x_1 = x_2 = \cdots = x_L \in D\}$ is an invariant distributional chaotic subset of F. Thus, the proof is ended.

Theorem 3.3 *For any* $0 < \varepsilon \leq 1$ *and any continuous selfmap* f *on* $[0, 1]$ *with positive topological entropy, the system* (1.1) *is* ω*-chaotic.*

Proof Applying Proposition 2.3, let D be an uncountable ω-scrambled subset of f and $\mathscr{D} := \{(x_1, \ldots, x_L \in I^L : x_1 = \cdots = x_L \in D\}$. For any $\overrightarrow{x} = (x, \ldots, x) \in \Delta$, it is not difficult to check that

$$\omega_F(\overrightarrow{x}) = \Big\{(y_1, \ldots, y_L) \in I^L : y_1 = \cdots = y_L \in \omega_f(x)\Big\}.$$

For any $(x, \ldots, x) \in \mathrm{Per}(F) \cap \Delta$, we have $x \in \mathrm{Per}(f)$. This implies that

$$\mathrm{Per}(F) \cap \Delta = \Big\{(x_1, \ldots, x_L) \in I^L : x_1 = \cdots = x_L \in \mathrm{Per}(f)\Big\}.$$

As $\omega_F(\overrightarrow{x}) \subset \Delta$, it follows that

$$\begin{aligned}\omega_F(\overrightarrow{x}) \setminus \mathrm{Per}(F) &= \Delta \cap (\omega_F(\overrightarrow{x}) \setminus \mathrm{Per}(F)) \\ &= \big(\Delta \cap \omega_F(\overrightarrow{x})\big) \setminus (\Delta \cap \mathrm{Per}(F)) \\ &= \Big\{(y_1, \ldots, y_L) \in I^L : y_1 = \cdots = y_L \in \omega_f(x) \setminus \mathrm{Per}(f)\Big\}.\end{aligned}$$

So, it is easy to see that $\mathscr{D}$ is a ω-scrambled set of F. Hence, F is ω-chaotic. Thus, the proof is complete.

It follows directly from Theorems 3.2, 3.3, and [20, Theorem 3] that the following is true. □

Theorem 3.4 *For any* $0 < \varepsilon \leq 1$ *and any continuous selfmap* f *on* $[0, 1]$ *with positive topological entropy, the system* (1.1) *is* $\mathscr{P}$*-chaotic. Consequently, the system* (1.1) *is* $\mathscr{P}$*-chaotic for each unimodal map* f.

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